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# Approximate External Boundaries for Truncated Models of Unbounded Media

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The objective of this study is to obtain simple boundary springs that when used in conjunction with Lysmer's dampers could result in a simple and effective artificial boundary for soil island models of soil-structure interaction problems. In particular, simple approximate expressions for the stiffnesses of distributed boundary springs appropriate for hemispherical, cylindrical, and rectangular soil islands are derived. Numerical values for the stiffness coefficients are presented for different aspect ratios of the soil islands.

## 1. INTRODUCTION

A variety of dynamic soil-structure interaction problems involving foundations with complex geometries or irregular soil deposits are analyzed by use of a soil island approach in which a portion of the soil surrounding the foundation is modeled, usually in a discretized fashion, together with the foundation and the superstructure. This convenient and flexible approach requires imposing some appropriate boundary conditions on the external artificial boundary of the soil island so that the response of the truncated soil model approaches that of an unbounded medium. While effective, quiet, non-reflecting or transmitting boundaries have been developed for this purpose, they are typically not used in the initial stages of model development and analysis as these special boundaries are not implemented in some of the most commonly used computational codes.

The use of Lysmer's dampers on the artificial boundary has emerged as a simple and effective way to reduce unwanted reflections from the boundary and to stabilize the overall model in the dynamic case. However, a low frequency drift of the overall model may be obtained if additional boundary springs are not introduced.

The objective of this study is to obtain simple boundary springs that when used in conjunction with Lysmer's dampers could result in an effective artificial boundary for a

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wider frequency range. Lysmer's dampers are local devices that relate components of the traction vector at a point with components of the velocity vector at the same point through damper constants per unit area that depend on the density of the soil and on the velocities of P-and S-waves also at the same point. Dimensional analysis indicates that if springs are introduced relating traction components to displacement components at a point, the resulting spring constants per unit area must depend on the elastic moduli and on some length scale. This length scale is related to the overall size of the soil island and thus reflects the global problem.

The spring constants also reflect some global characteristics such as the relative importance of resultant forces and resultant moments acting on the foundation.

In this paper, approximate boundary springs for hemispherical, cylindrical, and rectangular soil islands are obtained by consideration of the far-field displacements and stresses for static forces acting on an unbounded medium. The classical static solutions of Kelvin (Thomson, 1848,1882), Boussinesq (1878, 1885), and Cerruti (1882) are used to derive simple expressions and numerical values for the spring constants per unit area for these three soil island geometries. The emphasis is on simplicity for practical applications, and for this reason, a number of approximations are introduced.

The displacements and stresses resulting from applied moments decay more quickly in the far-field than the corresponding quantities for applied forces. For this reason, the rocking and torsional response is less sensitive to the location of the external boundary than the response to applied forces. The boundary springs presented herein are based on the response to resultant forces, which are more affected by the presence of the artificial boundary. The resulting springs give excellent results for the translational response but may not be sufficiently accurate to represent predominantly rocking or torsional motions when the artificial boundary is placed extremely close to the foundation. For reasonably sized soil islands, the obtained springs should provide good results for both the translation and rotational components of the response of the foundation.

## 2. HEMISPHERICAL REGION

To begin with, we consider the case of a hemispherical soil island or truncated elastic region of radius $R = a$. Referring to spherical coordinates $(R, \theta, \varphi)$, the boundary conditions on the artificial boundary $R = a$ are approximated by

$$\sigma_{RR}\left(a,\theta,\varphi\right) = -k_{RR}u_R\left(a,\theta,\varphi\right) \tag{2.1a}$$

$$\sigma_{R\theta}\left(a,\theta,\varphi\right) = -k_{R\theta}u_\theta\left(a,\theta,\varphi\right) \tag{2.1b}$$

$$\sigma_{R\varphi}\left(a,\theta,\varphi\right) = -k_{R\varphi}u_\varphi\left(a,\theta,\varphi\right) \tag{2.1c}$$

where $k_{RR}$, $k_{R\theta}$, and $k_{R\varphi}$ represent appropriate spring constants per unit area. Estimates for these spring constants are obtained by rewriting Eqs. (2.1a, b, c) in the form

$$k_{RR} = -\sigma_{RR}/u_R \ , \qquad k_{R\theta} = -\sigma_{R\theta}/u_\theta \ , \qquad k_{R\varphi} = -\sigma_{R\varphi}/u_\varphi \tag{2.2a, b, c}$$

and by calculating the stress-to-displacement ratios on the right-hand-side by use of some simple exact solutions for an unbounded medium and for a half-space. In some cases, the resulting estimates of $k_{RR}$, $k_{R\theta}$, and $k_{R\varphi}$ may be functions of $\theta$ and $\varphi$. In these cases, it may be convenient for practical applications to use the average estimates given by

$$\bar{k}_{RR} = -\bar{\sigma}_{RR}/\bar{u}_R \ , \qquad \bar{k}_{R\theta} = -\bar{\sigma}_{R\theta}/\bar{u}_\theta \ , \qquad \bar{k}_{R\varphi} = -\bar{\sigma}_{R\varphi}/\bar{u}_\varphi \tag{2.3a, b, c}$$

where $\bar{\sigma}_{R\alpha}$ and $\bar{u}_\alpha \left(\alpha = r,\theta,\varphi\right)$ represent weighted averages of the tractions and displacements over the artificial boundary.

### 2.1 BOUNDARY SPRINGS BASED ON SOLUTIONS FOR AN UNBOUNDED MEDIUM

It is convenient to start with Kelvin's solution obtained in 1848 (Thomson, 1882) for the response of an unbounded elastic medium subjected to a concentrated vertical force $P_z$ acting at the origin of the coordinate system. The quantities of interest, in spherical coordinates $\left(R,\theta,\varphi\right)$, are (Love, 1944, p. 202):

$$\left(u_R, u_\theta, u_\varphi\right) = \frac{P_z}{4\pi\mu R}\left(\sin\varphi,\ 0,\ \frac{\left(3-4\nu\right)}{4\left(1-\nu\right)}\cos\varphi\right) \tag{2.4a}$$

$$\left(\sigma_{RR}, \sigma_{R\theta}, \sigma_{R\varphi}\right) = -\frac{P_z}{4\pi R^2}\left(\frac{\left(2-\nu\right)}{\left(1-\nu\right)}\sin\varphi,\ 0,\ \frac{\left(1-2\nu\right)}{2\left(1-\nu\right)}\cos\varphi\right) \tag{2.4b}$$

where $\mu$ denotes the shear modulus, $\nu$ the Poisson's ratio, and $\varphi$ the latitude measured from the plane $z = 0 \left(x = R\cos\varphi\cos\theta,\ y = R\cos\varphi\sin\theta,\ z = R\sin\varphi\right)$. Substitution from

Eqs. (2.4a), and (2.4b) into Eqs. (2.2) leads to the following estimates for the spring constants $k_{RR}$ and $k_{R\varphi}$:

$$k_{RR}^{K} = \frac{(2-\nu)}{(1-\nu)}\frac{\mu}{R} \tag{2.5}$$

$$k_{R\varphi}^{K} = \frac{2(1-2\nu)}{(3-4\nu)}\frac{\mu}{R} \tag{2.6}$$

which are independent of $\theta$ and $\varphi$ and depend only on the radius $R = a$ of the artificial boundary.

The spherical symmetry of the problem suggests that $k_{R\theta}$ should be equal to $k_{R\varphi}$ for the unbounded medium. This can be confirmed by considering the solution for a concentrated horizontal force $P_x$ acting at the origin. Again, the quantities of interest, in spherical coordinates, are (Love, 1944, p. 202):

$$\left(u_R, u_\theta, u_\varphi\right) = \frac{P_x}{4\pi\mu R}\left(\cos\varphi\,\cos\theta,\ -\frac{(3-4\nu)}{4(1-\nu)}\sin\varphi,\ -\frac{(3-4\nu)}{4(1-\nu)}\sin\varphi\cos\theta\right) \tag{2.7a}$$

$$\left(\sigma_{RR}, \sigma_{R\theta}, \sigma_{r\varphi}\right) = \frac{P_x}{4\pi R^2}\left[-\frac{(2-\nu)}{(1-\nu)}\cos\varphi\cos\theta,\ \frac{(1-2\nu)}{2(1-\nu)}\sin\theta, \frac{(1-2\nu)}{2(1-\nu)}\sin\varphi\cos\theta\right] \tag{2.7b}$$

Substitution from Eqs. (2.7a) and (2.7b) into Eqs. (2.2) leads to the same expressions for $k_{RR}^{K}$ and $k_{R\varphi}^{K}$ given by Eqs. (2.5) and (2.6), respectively. The new result is

$$k_{R\theta}^{K} = \frac{2(1-2\nu)}{(3-4\nu)}\frac{\mu}{R} \tag{2.8}$$

which confirms that $k_{R\theta} = k_{R\varphi}$ for this case.

The boundary conditions (2.1a, b, c) in conjunction with the springs constants $k_{RR}^{K}$, $k_{R\varphi}^{K}$, and $k_{R\theta}^{K}$ given by Eqs. (2.5), (2.6), and (2.8), are exact for concentrated forces acting at the origin. The next step is to consider the boundary conditions and spring constants for concentrated moments acting also at the origin. The simplest case corresponds to a vertical a torque $T_z$ acting at $R = 0$. The displacement and stress components of interest, in this case, are given by

$$u_\theta = \frac{T_z}{8\pi\mu}\frac{\cos\varphi}{R^2} \quad , \qquad \sigma_{R\theta} = -\frac{3T_z}{8\pi}\frac{\cos\varphi}{R^3} \tag{2.9a, b}$$

which lead to the alternative estimate of the spring constant $k_{R\theta}$ on the surface $R = a$:

$$k_{R\theta}^{T} = 3\mu/R \tag{2.10}$$

Based on the spherical symmetry, it is expected that consideration of a concentrated moment $M_x$ about the x-axis will lead to

$$k_{R\varphi}^{M} = 3\mu/R \tag{2.11}$$

The spring constants $K_{R\theta}^{T} = K_{R\varphi}^{M}$ are independent of $\theta$ and $\varphi$ and are considerably larger than those given by Eqs. (2.6) and (2.8). For example, for $\nu = 1/4$, $k_{R\theta}^{K} = k_{R\varphi}^{K} = (1/2)(\mu/R)$ while $k_{R\theta}^{T} = K_{R\varphi}^{M} = 3(\mu/R)$.

Since the displacements and stresses for a concentrated torque or moment decay very quickly with distance ($R^{-2}$ and $R^{-3}$, respectively), the effects of an artificial boundary on the response to these excitations are much less pronounced than those for resultant forces for which the response decays more slowly with ($R^{-1}$ and $R^{-2}$, respectively). This suggests that the estimates of $k_{RR}$, $k_{R\varphi}$, and $k_{R\theta}$ based on resultant forces are typically more pertinent than those based on concentrated moments.

The boundary spring constants derived in this section are based on the fundamental solutions for an unbounded full-space. Estimates of the corresponding stiffnesses for an elastic half-space are considered next.

## 2.2 ESTIMATES OF $k_{RR}$ FOR A HALF-SPACE

An estimate for $k_{RR}$ can be obtained by consideration of Cerruti's solution (1882) for a horizontal point load $P_x$ applied to the surface of a half-space $(z > 0)$. The quantities of interest are

$$u_R = \frac{P_x}{2\pi\mu R}\left[2(1-\nu) - \frac{(1-2\nu)}{1+\sin\varphi}\right]\cos\varphi\cos\theta \tag{2.12a}$$

$$\sigma_{RR} = -\frac{P_x}{\pi R^2}\left[(2-\nu) - \frac{(1-2\nu)}{1+\sin\varphi}\right]\cos\varphi\cos\theta \tag{2.12b}$$

which lead to

$$k_{RR}^{C}(\varphi) = k_{RR}^{C}(0)\left[\frac{1+\left(\dfrac{2-\nu}{1+\nu}\right)\sin\varphi}{1+2(1-\nu)\sin\varphi}\right] \tag{2.13}$$

where $k_{RR}^{C}(0) = 2(1+\nu)\mu/R$. For $\nu = 0$ and $\nu = 1/2$, the quantity within the square brackets in Eq. (2.13) is equal to one, and consequently, the spring constant $k_{RR}^{C}$ becomes independent of $\varphi$ and equal to $k_{RR}^{C}(0)$ for these two values of $\nu$. The stiffness $k_{RR}^{C}(\varphi)$ varies from $k_{RR}^{C}(0)$ for $\varphi = 0$ to $k_{RR}(\pi/2) = \left[6/(3-2\nu)\right](\mu/R)$ for $\varphi = \pi/2$.

Since the variation of $k_{RR}^{C}$ with $\varphi$ is not very strong, it is convenient to use a constant value estimated as $\bar{k}_{RR}^{C} = -\bar{\sigma}_{RR}^{C}/\bar{u}_{R}^{C}$ where

$$\bar{\sigma}_{RR}^{C} = \frac{2}{\pi}\int_0^{\pi/2}\sigma_{RR}\,d\varphi = -\frac{P_x}{\pi R^2}\cdot\frac{2}{\pi}\left[(2-\nu)-(1-2\nu)\ln 2\right]\cos\theta \tag{2.14a}$$

$$\bar{u}_{R}^{C} = \frac{2}{\pi}\int_0^{\pi/2}u_R\,d\varphi = \frac{P_x}{2\pi\mu R^2}\cdot\frac{2}{\pi}\left[2(1-\nu)-(1-2\nu)\ln 2\right]\cos\theta \tag{2.14b}$$

The resulting estimate for $\bar{k}_{RR}^{C}$ is

$$\bar{k}_{RR}^{C} = 2\left[\frac{(2-\nu)-(1-2\nu)\ln 2}{2(1-\nu)-(1-2\nu)\ln 2}\right]\frac{\mu}{R} \tag{2.15}$$

which can be approximated by

$$\bar{k}_{RR}^{C} = 2\cdot\left(\frac{13+4\nu}{13-6\nu}\right)\frac{\mu}{R} \tag{2.16}$$

This estimate $\bar{k}_{RR}^{C}$ of the radial spring is numerically equal to $k_{RR}^{K}$ for $\nu = 0$ and $\nu = 1/2$ and is similar to $k_{RR}^{K}$ for other values of $\nu$.

The solution of Boussinesq (1885) obtained in 1878 for a vertical point load applied on the surface of the half-space $z > 0$ can also be used to obtain an estimate of $k_{RR}$. Again, using average values for $\sigma_{RR}$ and $u_r$ leads to

$$\bar{k}_{RR}^{B} = 2\left[\frac{(2-\nu)-(1-2\nu)(\pi/4)}{2(1-\nu)-(1-2\nu)(\pi/4)}\right]\frac{\mu}{R} \tag{2.17}$$

which matches $\bar{k}_{RR}^{C}$ for $\nu = 0$ and $\nu = 1/2$ and is similar for other values of $\nu$.

## 2.3 ESTIMATES OF $k_{R\theta}$ FOR A HALF-SPACE

An estimate for $k_{R\theta}$ can be obtained by considering the response of a half-space to a horizontal point force $P_x$ acting at the origin. Cerruti's solution (1882) indicates that

$$u_{\theta} = -\frac{P_x}{4\pi\mu R}\left[1+\frac{(1-2\nu)}{1+\sin\varphi}\right]\sin\theta \tag{2.18a}$$

$$\sigma_{R\theta} = \frac{P_x}{2\pi R^2}\frac{(1-2\nu)\cos^2\varphi}{(1+\sin\varphi)^2}\sin\theta \tag{2.18b}$$

from where the estimate

$$k_{R\theta}^{C}(\varphi) = \left[\frac{2(1-2\nu)(1-\sin\varphi)}{2(1-\nu)+\sin\varphi}\right]\frac{\mu}{R} \tag{2.19}$$

of the stiffness $k_{R\theta}$ on the surface $R = a$ is obtained. The stiffness $k_{R\theta}^{C}$ is a function of $\varphi$ and varies from $\left[(1-2\nu)/(1-\nu)\right]\mu/R$ on the surface $\varphi = 0$ to a value of zero on the axis $\varphi = \pi/2$. An equivalent uniform stiffness $\bar{k}_{R\theta}^{C}$ can be obtained as $\bar{k}_{R\theta}^{C} = -\bar{\sigma}_{R\theta}^{C}/\bar{u}_{\theta}^{C}$ where

$$\bar{\sigma}_{r\theta}^{C} = \frac{2}{\pi}\int_0^{\pi/2}\sigma_{r\theta}\,d\varphi = \frac{P_x}{4\pi R^2}\left[\,0.546(1-2\nu)\right] \tag{2.20a}$$

$$\bar{u}_{\theta}^{C} = \frac{2}{\pi}\int_0^{\pi/2}u_{\theta}\,d\varphi = -\frac{P_x}{4\pi\mu R}\left[1+0.637(1-2\nu)\right]\;. \tag{2.20b}$$

The resulting stiffness can be approximated by

$$\bar{k}_{R\theta}^{C} = \frac{3(1-2\nu)}{(9-7\nu)}\frac{\mu}{R} \tag{2.21}$$

which is a fraction ranging from $0.5(\nu = 0)$ to $0.27(\nu = 1/2)$ of the stiffness $k_{R\theta}^{K}$ for an unbounded medium.

### 2.4 ESTIMATES OF $k_{R\varphi}$ FOR A HALF-SPACE

An estimate for $k_{R\varphi}$ can be obtained by use of Boussinesq's solution (1885) for a normal point load $P_z$ acting on the surface of the half-space $z > 0$. The quantities of interest, in this case, are:

$$u_{\varphi} = \frac{P_z}{4\pi\mu R}\left[\frac{(3-4\nu)\sin\varphi + 2(1-\nu)}{1+\sin\varphi}\right]\cos\varphi \tag{2.22a}$$

$$\sigma_{R\varphi} = -\frac{P_z}{2\pi R^2}\left[\frac{(1-2\nu)\sin\varphi\cos\varphi}{1+\sin\varphi}\right] \tag{2.22b}$$

and the resulting estimate of $k_{R\varphi}$ is

$$k_{R\varphi}^{B}(\varphi) = \frac{2(1-2\nu)}{5-6\nu}\frac{\mu}{R}\left[\frac{\sin\varphi}{1-\left(\dfrac{3-4\nu}{5-6\nu}\right)(1-\sin\varphi)}\right] \tag{2.23}$$

The stiffness $k_{R\varphi}^{B}$ varies from $k_{R\varphi}^{B} = 0$ on the surface $\varphi = 0$ to $k_{R\varphi}^{B} = \left[2(1-2\nu)/(5-6\nu)\right](\mu/R)$ on the axis $\varphi = \pi/2$. It is apparent that the shear spring constant $k_{R\varphi}$, defined by Eq. (2.23), vanishes for all angles $\varphi$ for $\nu = 1/2$.

For practical applications, it may be convenient to use a stiffness $k_{R\varphi}$ independent of $\varphi$. This constant stiffness $\bar{k}_{R\varphi}$ can be estimated as $\bar{k}_{R\varphi}^{B} = -\bar{\sigma}_{R\varphi}^{B}/\bar{u}_{\varphi}^{B}$ where

$$\bar{\sigma}_{R\varphi}^{B} = \frac{2}{\pi}\int_0^{\pi/2}\sigma_{R\varphi}\,d\varphi = -\frac{P_z}{2\pi R^2}(1-2\nu)\frac{2}{\pi}(1-\ln 2) \tag{2.24a}$$

$$\bar{u}_{\varphi}^{B} = \frac{2}{\pi}\int_0^{\pi/2}u_{\varphi}\,d\varphi = \frac{P_z}{4\pi\mu R}\frac{2}{\pi}\left[(3-4\nu)-(1-2\nu)\ln 2\right] \tag{2.24b}$$

The resulting estimate of $k_{R\varphi}^{B}$ is

$$\bar{k}_{R\varphi}^{B} = \frac{\mu}{R}\left[\frac{2(1-2\nu)(1-ln\,2)}{(3-4\nu)-(1-2\nu)ln\,2}\right] \tag{2.25}$$

which can be approximated by

$$\bar{k}_{R\varphi}^{B} = \frac{4(1-2\nu)}{(15-17\nu)}\frac{\mu}{R} \tag{2.26}$$

The average stiffness $\bar{k}_{R\varphi}^{B}$ for the half-space is also a fraction of the stiffness $k_{R\varphi}^{K}$ for an unbounded medium.

## 2.5 SUMMARY AND COMPARISONS

In summary, for practical applications, the following values for the stiffnesses $k_{RR}$, $k_{R\theta}$, and $k_{R\varphi}$ for uniform boundary springs are suggested:

(a) full-space

$$k_{RR} = \frac{(2-\nu)}{(1-\nu)}\frac{\mu}{R} \quad , \qquad k_{R\theta} = \frac{2(1-2\nu)}{(3-4\nu)}\frac{\mu}{R} \quad , \qquad k_{R\varphi} = \frac{2(1-2\nu)}{(3-4\nu)}\frac{\mu}{R} \tag{2.27a, b, c}$$

(b) half-space

$$k_{RR} = 2\frac{(13+4\nu)}{(13-6\nu)}\frac{\mu}{R} \quad , \qquad k_{R\theta} = \frac{3(1-2\nu)}{(9-7\nu)}\frac{\mu}{R} \quad , \qquad k_{R\varphi} = \frac{4(1-2\nu)}{(15-17\nu)}\frac{\mu}{R} \tag{2.28a, b, c}$$

Numerical values for the stiffnesses $k_{RR}$, $k_{R\theta}$, and $k_{R\varphi}$ for some typical Poisson's ratios are presented in Table 2.1.

An extreme and not very realistic test of the boundary springs given by Eqs. (2.27) and (2.28) can be obtained by applying these springs directly to the boundary of a rigid hemispherical foundation of radius $a$ embedded in an elastic half-space. The vertical $K_{VV}$, horizontal $K_{HH}$, coupling $K_{HM}$, rocking $K_{MM}$, and torsional $K_{TT}$ static impedance functions for such a foundation resting on uniform springs are given by

$$K_{VV} = \frac{2\pi}{3}\left(k_{RR} + 2k_{R\varphi}\right)a^2 \quad , \qquad K_{HH} = \frac{\pi}{3}\left(2k_{RR} + 3k_{R\theta} + k_{R\varphi}\right)a^2 \tag{2.29}$$

and

$$K_{HM} = K_{MH} = \frac{\pi}{2}\left(k_{R\theta} + k_{R\varphi}\right)a^3 \ , \quad K_{MM} = \frac{\pi}{3}\left(k_{R\theta} + 3k_{R\varphi}\right)a^4 , \quad K_{TT} = \frac{4\pi}{3}k_{R\theta}a^4 \ . \tag{2.30}$$

Substitution from Eqs. (2.29) and (2.30) into Eq. (2.31) leads to the numerical values for the impedances listed in Table 2.2 for the case $\nu = 1/4$. The resulting crude estimates of the impedance functions are compared with more accurate numerical results obtained by Luco and Wong (1986) by use of an indirect boundary integral method.

**Table 2.1** Normalized Stiffnesses for Boundary Spring

| | $\nu = 0$ | $\nu = 1/4$ | $\nu = 1/3$ | $\nu = 1/2$ |
|---|---|---|---|---|
| Full-Space | | | | |
| $Rk_{RR}/\mu$ | 2.000 | 2.333 | 2.500 | 3.000 |
| $Rk_{R\theta}/\mu$ | 0.667 | 0.500 | 0.400 | 0.000 |
| $Rk_{R\varphi}/\mu$ | 0.667 | 0.500 | 0.400 | 0.000 |
| Half-Space | | | | |
| $Rk_{RR}/\mu$ | 2.000 | 2.435 | 2.601 | 3.000 |
| $Rk_{R\theta}/\mu$ | 0.333 | 0.207 | 0.150 | 0.000 |
| $Rk_{R\varphi}/\mu$ | 0.267 | 0.186 | 0.143 | 0.000 |

The comparison in Table 2.2 indicates that the boundary springs based on solutions for a half-space [Eq. (2.28)] when applied directly to the boundary of a rigid hemispherical foundation lead to reasonable estimates for the vertical (-18% error) and horizontal (-25% error) static impedance functions. Slightly better results for $K_{VV}$ and $K_{HH}$ are obtained in this extreme case by use of the spring constants [Eq. (2.27)] based on Kelvin's solution for a full-space (-3% error for $K_{VV}$ and 18% error for $K_{HH}$). These results suggest that the boundary springs derived here will substantially improve the response of even a small soil island.

**Table 2.2** Comparison of Impedance Functions for a Hemispherical Foundation $(\nu = 1/4)$

| | $K_{VV}/\mu a$ | $K_{HH}/\mu a$ | $K_{HM}/\mu a^2$ | $K_{MM}/\mu a^3$ | $K_{TT}/\mu a^3$ |
|---|---|---|---|---|---|
| Eq. (2.28) | 5.88 | 5.94 | 0.62 | 0.80 | 0.87 |
| Eq. (2.27) | 6.98 | 9.42 | 1.57 | 2.09 | 2.09 |
| Luco&Wong | 7.19 | 7.95 | 3.98 | 10.34 | 12.57 |

The comparisons in Table 2.2 show large errors for the impedance functions involving rocking and torsional moments when the boundary springs are applied directly to the foundation. This is to be expected as the boundary springs were based on resultant forces and not moments. However, a moderately-sized soil island should be sufficient to obtain accurate results for the quickly attenuating response to applied moments.

## 3. CYLINDRICAL REGIONS

Next, we consider the selection of boundary springs so that a finite elastic cylindrical region of radius $a$ and depth $h$ can approximately represent an elastic half-space $(z > 0)$. Cylindrical coordinates $(r,\theta,z)$ such that $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$ will be used to determine the boundary springs.

**Lateral Boundary.** The boundary conditions on the lateral boundary $r = a$ $(0 < \theta < 2\pi\,,\ 0 < z < h)$ of the cylinder are approximated by

$$\sigma_{rr} = -k_{rr}u_r\ , \qquad \sigma_{r\theta} = -k_{r\theta}u_\theta\ , \qquad \sigma_{rz} = -k_{rz}u_z \tag{3.1}$$

where $k_{rr}$, $k_{r\theta}$, and $k_{rz}$ represent the stiffness per unit area of springs distributed over the lateral boundary of the cylinder. These constants are estimated by use of:

$$k_{r\alpha}(a,\theta,z) = -\sigma_{r\alpha}(a,\theta,z)/u_\alpha(a,\theta,z)\ , \qquad (\alpha = r,\theta,z) \tag{3.2}$$

where $u_r$, $u_\theta$, $u_z$ and $\sigma_{rr}$, $\sigma_{r\theta}$, $\sigma_{rz}$ are the displacement and stress components for known fundamental solutions for a half-space. For practical applications, it may be convenient to use spring constants that are independent of position. Average estimates $\bar{k}_{r\alpha}\left(\alpha=r,\theta,z\right)$ can be written in the form

$$\bar{k}_{r\alpha} = \mu\,\beta_{r\alpha}/a \tag{3.3}$$

where the dimensionless stiffnesses $\beta_{r\alpha}$ are given by:

$$\beta_{r\alpha} = -\int_0^h\int_0^{2\pi}\left(\sigma_{r\alpha}/\mu\right)\varpi_\alpha\left(\theta\right)a\,d\theta\,dz\Big/\int_0^h\int_0^{2\pi}\left(u_\alpha/a\right)\varpi_\alpha\left(\theta\right)a\,d\theta\,dz\,,\qquad \left(\alpha=r,\theta,z\right) \tag{3.4}$$

where $\varpi\left(\theta\right)$ is a weighting function.

**Basal Boundary.** The boundary conditions on the basal boundary $z=h$ $\left(0<r<a\,,\,0<\theta<2\pi\right)$ of the cylinder are approximated by

$$\sigma_{zr} = -k_{zr}u_r\ ,\qquad \sigma_{z\theta} = -k_{z\theta}u_\theta\ ,\qquad \sigma_{zz} = -k_{zz}u_z \tag{3.5}$$

where $k_{zr}$, $k_{z\theta}$, and $k_{zz}$ represent the stiffness per unit area of springs distributed over the base of the cylinder. These stiffnesses are obtained from

$$k_{z\alpha}\left(r,\theta,h\right) = -\sigma_{z\alpha}\left(r,\theta,h\right)\big/u_\alpha\left(r,\theta,h\right),\qquad \left(\alpha=r,\theta,h\right) \tag{3.6}$$

where again $u_\alpha$ and $\sigma_{z\alpha}$ are displacements and stresses from known solutions for a half-space. Average basal spring constants independent of position are given by

$$\bar{k}_{z\alpha} = \mu\,\beta_{z\alpha}/h\ ,\qquad \left(\alpha=r,\theta,z\right) \tag{3.7}$$

where

$$\beta_{z\alpha} = -\int_0^{2\pi}\int_0^a\left(\sigma_{z\alpha}/\mu\right)\varpi_\alpha\left(\theta\right)r\,dr\,d\theta\Big/\int_0^{2\pi}\int_0^a\left(u_\alpha/h\right)\varpi_\alpha\left(\theta\right)r\,dr\,d\theta\ ,\ \left(\alpha=r,\theta,z\right) \tag{3.8}$$

The stiffnesses $k_{rz}$ and $k_{zz}$ will result from consideration of Boussinesq's (1885) solution while $k_{rr}$, $k_{r\theta}$, $k_{zr}$, and $k_{z\theta}$ will be obtained from Cerruti's solution (1882).

## 3.1 ESTIMATES OF THE SPRING CONSTANTS $k_{zz}$ AND $k_{rz}$

To obtain the stiffnesses $k_{zz}$ and $k_{rz}$ of the boundary springs for a cylindrical boundary, we consider the solution of Boussinesq (1885), later complemented by Terazawa (1916), for a vertical point load $P_z$ acting on the surface of an elastic half-space $\left(z>0\right)$. In cylindrical coordinates $\left(r,\theta,z\right)$, the displacement and stress components of interest are:

$$u_z=\frac{P_z}{4\pi\mu}\left[\frac{z^2}{R^3}+\frac{2\left(1-\nu\right)}{R}\right]\quad,\qquad \sigma_{rz}=-\frac{3P_z}{2\pi}\frac{rz^2}{R^5}\quad,\qquad \sigma_{zz}=-\frac{3P_z}{2\pi}\frac{z^3}{R^5}\tag{3.9a,b,c}$$

where $\mu$ is the shear modulus, $\nu$ is Poisson's ratio and $R=\sqrt{r^2+z^2}$.

At the base $z=h$ of the cylinder $\left(0<r<a\right)$, we obtain

$$k_{zz}\left(r,h\right)=\frac{6\mu}{h}\frac{\left(h/R\right)^4}{\left[\left(h/R\right)^2+2\left(1-\nu\right)\right]}\tag{3.10}$$

where $R=\sqrt{r^2+h^2}$. The stiffness $k_{zz}$ is independent of $\theta$ but varies with $r$ from the value $k_{zz}\left(0,h\right)=6\mu/\left(3-2\nu\right)h$ on the axis $r=0$ to the limit $k_{zz}\left(r,h\right)\rightarrow\left[3\mu/\left(1-\nu\right)h\right]\left(h/r\right)^4$ as $r\rightarrow\infty$. The representative constant stiffness $\overline{k}_{zz}=\mu\beta_{zz}/h$ is determined by

$$\beta_{zz}=\frac{6h^4\int_0^a rR^{-5}dr}{\int_0^a\left[\left(h/R\right)^2+2\left(1-\nu\right)\right]rR^{-1}dr}\tag{3.11}$$

where $R=\sqrt{r^2+h^2}$. The dimensionless stiffness $\beta_{zz}$ is only a function of $h/a$ and $\nu$. Numerical values for $\beta_{zz}$ for $\nu=1/3$ are shown in Fig. 3.1 as a function of $h/a$.

On the mantle $r=a$ of the cylinder, we obtain

$$k_{rz}\left(a,z\right)=\frac{6\mu}{a}\frac{\left(z/R\right)^2\left(a/R\right)^2}{\left[\left(z/R\right)^2+2\left(1-\nu\right)\right]}\tag{3.12}$$

where $R=\sqrt{a^2+z^2}$. The stiffness $k_{rz}(a,z)$ tends to zero as $z\to 0$ and $k_{rz}(a,z)\to\left[6\mu/(3-2\nu)a\right](a/z)^2$ for $z\to\infty$. The representative constant stiffness $\overline{k}_{rz}=\mu\beta_{rz}/a$ is determined by

$$\beta_{rz}=\frac{6a^2\int_0^h z^2R^{-5}dz}{\int_0^h\left[(z/R)^2+2(1-\nu)\right]R^{-1}dz} \tag{3.13}$$

where $R=\sqrt{z^2+a^2}$. Numerical values of $\beta_{rz}$ for $\nu=1/3$ are shown in Fig. 3.1 as a function of $h/a$.

### 3.2 ESTIMATES OF THE SPRING CONSTANTS $k_{rr}$, $k_{r\theta}$, $k_{zr}$, AND $k_{z\theta}$

Estimates of the spring constants $k_{rr}$, $k_{r\theta}$, $k_{zr}$, and $k_{z\theta}$ can be obtained from Cerruti's (1882) solution for a horizontal point load $P_z$ on the surface of an elastic half-space $(z>0)$. The quantities of interest are:

$$u_r=\frac{P_x}{4\pi\mu R}\left\{1+\frac{r^2}{R^2}+(1-2\nu)\left[\frac{R}{R+z}-\frac{r^2}{(R+z)^2}\right]\right\}\cos\theta \tag{3.14A}$$

$$u_\theta=-\frac{P_x}{4\pi\mu R}\left\{1+(1-2\nu)\left(\frac{R}{R+z}\right)\right\}\sin\theta \tag{3.14b}$$

and

$$\sigma_{rr}=-\frac{P_x}{2\pi R^2}\left\{\frac{3r^2}{R^2}-(1-2\nu)\frac{R^2}{(R+z)^2}\right\}\frac{r}{R}\cos\theta \tag{3.15a}$$

$$\sigma_{r\theta}=\frac{P_x}{2\pi R^2}(1-2\nu)\frac{rR^2}{(R+z)^2}\sin\theta \tag{3.15b}$$

$$\sigma_{zr}=-\frac{P_x}{2\pi R^2}\frac{3r^2z}{R^3}\cos\theta \tag{3.15c}$$

$$\sigma_{z\theta}=0 \tag{3.15d}$$

where $R=\sqrt{r^2+z^2}$.

The resulting spring constants $k_{rr}$ and $k_{r\theta}$ on the lateral boundary $r=a\ (0<\theta<2\pi, 0<z<h)$ are given by

$$k_{rr}(a,z)=\frac{2\mu}{R}\frac{\left\{\frac{3r^2}{R^2}-(1-2\nu)\frac{R^2}{(R+z)^2}\right\}\frac{r}{R}}{\left\{1+\frac{r^2}{R^2}+(1-2\nu)\left[\frac{R}{R+z}-\frac{r^2}{(R+z)^2}\right]\right\}}\ ,\qquad (r=a)\qquad (3.16a)$$

$$k_{r\theta}(a,z)=\frac{2(1-2\nu)\mu}{R}\frac{\frac{rR}{(R+z)^2}}{\left\{1+(1-2\nu)\left(\frac{R}{R+z}\right)\right\}}\ ,\qquad (r=a)\qquad (3.16b)$$

where $R=\sqrt{a^2+z^2}$. The spring constant $k_{rr}$ and $k_{r\theta}$ depend on depth $z$. In particular, $k_{rr}\to 2(1+\nu)\mu/a$ and $k_{r\theta}=\left[(1-2\nu)/(1-\nu)\right]\mu/a$ as $z\to 0$.

The corresponding estimates of the spring constants $k_{zr}$ and $k_{z\theta}$ on the base $z=h$ of the cylindrical region are

$$k_{zr}(r,h)=\frac{6\mu}{R}\frac{\left(r^2z/R^3\right)}{\left\{1+\frac{r^2}{R^2}+(1-2\nu)\left[\frac{R}{R+z}-\frac{r^2}{(R+z)^2}\right]\right\}}\ ,\qquad (z=h)\qquad (3.17a)$$

$$k_{z\theta}(r,h)=0\qquad (3.17b)$$

where $R=\sqrt{r^2+h^2}$. The stiffness $k_{zr}$ depends on $r$, and, in particular, $k_{zr}\to(\mu/h)\left[12/(3-2\mu)\right](r/h)^2$ as $r\to 0$ and $k_{zr}\to(3\mu/h)(h/r)^2$ as $r\to\infty$. It is characteristic of Cerruti's solution that $\sigma_{z\theta}=0$ at all points in the half-space. Consequently, the resulting estimate of $k_{z\theta}$ is equal to zero.

Simpler, average spring constants $\bar{k}_{rr}$ and $\bar{k}_{r\theta}$ over the lateral boundary $(r=a)$ and $\bar{k}_{zr}$ and $\bar{k}_{z\theta}$ over the basal area $(z=h)$ can be written in the form

$$\bar{k}_{rr} = \mu\,\beta_{rr}/a \quad , \qquad\qquad \bar{k}_{r\theta} = \mu\,\beta_{r\theta}/a \tag{3.18}$$

and

$$\bar{k}_{zr} = \mu\,\beta_{zr}/h \quad , \qquad\qquad \bar{k}_{z\theta} = \mu\,\beta_{z\theta}/h \tag{3.19}$$

where

$$\beta_{rr} = \frac{2\int_0^h \left\{ \dfrac{3r^2}{R^2} - (1-2\nu)\dfrac{R^2}{(R+z)^2} \right\} \dfrac{r^2}{R^3}\, dz}{\int_0^h \left\{ 1 + \dfrac{r^2}{R^2} + (1-2\nu)\left[ \dfrac{R}{R+z} - \dfrac{r^2}{(R+z)^2} \right] \right\} \dfrac{dz}{R}} \quad , \qquad (r=a) \tag{3.20a}$$

$$\beta_{r\theta} = \frac{2(1-2\nu)\int_0^h r^2 R^{-1}\left(R+z^2\right)^{-2} dz}{\int_0^h \left\{ 1 + (1-2\nu)\left( \dfrac{R}{R+z} \right) \right\} \dfrac{dz}{R}} \quad , \qquad (r=a) \tag{3.20b}$$

and

$$\beta_{zr} = \frac{6\int_0^a r^3 z\, R^{-5} dr}{\int_0^a \left\{ 1 + \dfrac{r^2}{R^2} + (1-2\nu)\left[ \dfrac{R}{R+z} - \dfrac{r^2}{(R+z)^2} \right] \right\} \dfrac{r dr}{zR}} \quad , \qquad (z=h) \tag{3.21a}$$

$$\beta_{z\theta} = 0 \tag{3.21b}$$

The dimensionless stiffness coefficients $\beta_{rr}$, $\beta_{r\theta}$, and $\beta_{zr}$ depend only on $\nu$ and $h/a$. Numerical values for these coefficients are shown in Fig. 3.1 as a function of $h/a$ for $\nu = 1/3$. It should be noted that $\beta_{rr} \to 2(1+\nu)$ and $\beta_{r\theta} \to (1-2\nu)/(1-\nu)$ as $h/a \to 0$.

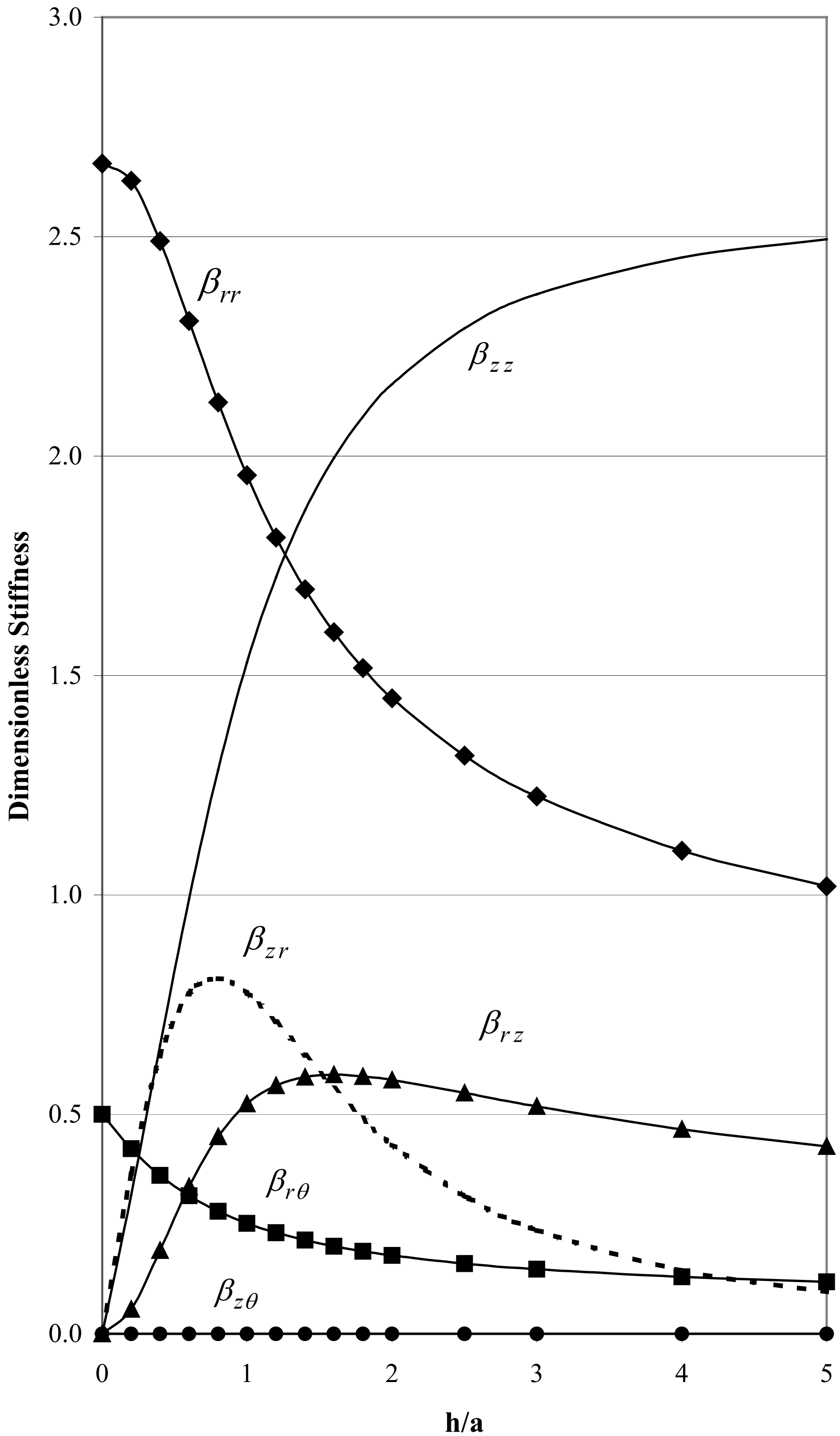


**Figure 3.1.** Normalized lateral $\left(\beta_{rr}, \beta_{r\theta}, \beta_{rz}\right)$ and basal $\left(\beta_{zr}, \beta_{z\theta}, \beta_{zz}\right)$ stiffness coefficients as a function of $h/a$ for a cylindrical soil island of radius $a$ and depth $h$ $\left(\nu = 1/3\right)$.

### 3.3 SUMMARY AND COMPARISON

In summary, average spring constants per unit area distributed over the mantle $(r = a)$ and base $(z = h)$ of the artificial boundary are given, respectively, by:

$$\bar{k}_{rr} = \mu\,\beta_{rr}/a \quad , \qquad \bar{k}_{r\theta} = \mu\,\beta_{r\theta}/a \quad , \qquad \bar{k}_{rz} = \mu\,\beta_{rz}/a \tag{3.22}$$

and

$$\bar{k}_{zr} = \mu\,\beta_{zr}/h \quad , \qquad \bar{k}_{z\theta} = \mu\,\beta_{z\theta}/h \quad , \qquad \bar{k}_{zz} = \mu\,\beta_{zz}/h \tag{3.23}$$

Numerical values for the dimensionless coefficients $\beta_{\alpha\beta}$ $(\alpha = r, z; \beta = r, \theta, z)$ for $\nu = 1/3$ are listed in Table 3.1.

**Table 3.1.** Numerical values for the normalized lateral $(\beta_{rr}, \beta_{r\theta}, \beta_{rz})$ and basal $(\beta_{zr}, \beta_{z\theta}, \beta_{zz})$ boundary stiffnesses $(\nu = 1/3)$.

| $h/a$ | $\beta_{rr}$ | $\beta_{r\theta}$ | $\beta_{rz}$ | $\beta_{zr}$ | $\beta_{z\theta}$ | $\beta_{zz}$ |
|---|---|---|---|---|---|---|
| 0.0 | 2.667 | 0.500 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.2 | 2.627 | 0.422 | 0.056 | 0.366 | 0.000 | 0.317 |
| 0.4 | 2.490 | 0.361 | 0.190 | 0.633 | 0.000 | 0.658 |
| 0.6 | 2.308 | 0.314 | 0.335 | 0.774 | 0.000 | 0.991 |
| 0.8 | 2.123 | 0.279 | 0.449 | 0.809 | 0.000 | 1.286 |
| 1.0 | 1.956 | 0.252 | 0.524 | 0.776 | 0.000 | 1.530 |
| 1.2 | 1.814 | 0.230 | 0.566 | 0.711 | 0.000 | 1.724 |
| 1.4 | 1.696 | 0.213 | 0.585 | 0.635 | 0.000 | 1.877 |
| 1.6 | 1.598 | 0.199 | 0.590 | 0.560 | 0.000 | 1.996 |
| 1.8 | 1.517 | 0.188 | 0.587 | 0.491 | 0.000 | 2.090 |
| 2.0 | 1.448 | 0.179 | 0.578 | 0.431 | 0.000 | 2.164 |
| 2.5 | 1.317 | 0.160 | 0.549 | 0.315 | 0.000 | 2.291 |
| 3.0 | 1.224 | 0.147 | 0.518 | 0.236 | 0.000 | 2.369 |
| 4.0 | 1.100 | 0.130 | 0.466 | 0.144 | 0.000 | 2.452 |
| 5.0 | 1.020 | 0.119 | 0.427 | 0.096 | 0.000 | 2.494 |

An extreme test of the spring constants obtained here for a cylindrical region, results from applying these springs directly to the boundary of a rigid cylindrical foundation of radius $\underline{a}$ and depth $\underline{h}$. The vertical $K_{VV}$ and horizontal $K_{HH}$ static impedance functions for the foundation for resting on the distributed springs are given by

$$K_{VV} = \mu a \left[ 2\pi \left( \frac{h}{a} \right) \beta_{rz} + \pi \left( \frac{a}{h} \right) \beta_{zz} \right] \tag{3.24a}$$

$$K_{HH} = \mu a \left[ \pi \left( \frac{h}{a} \right) (\beta_{rr} + \beta_{r\theta}) + \frac{\pi}{2} \left( \frac{a}{h} \right) (\beta_{zr} + \beta_{z\theta}) \right] \tag{3.24b}$$

which can be compared with the corresponding static impedance functions for a rigid cylindrical foundation embedded in an elastic half-space. A comparison of the estimates of the vertical and horizontal impedance functions given by Eqs. (3.24 a, b) with results obtained by Apsel and Luco (1987) for a cylindrical foundation embedded in a half-space is presented in Table 3.2. The differences range from 7 to 30 percent, which is remarkable considering that the boundary springs were derived under the assumption of a significant distance between the loaded area and the artificial boundary.

**Table 3.2.** Comparison of impedance functions for a rigid cylindrical foundation $(\nu = 1/4)$.

| $h/a$ | $K_{VV}$ Eq. (3.24a) | $K_{VV}$ Apsel & Luco | $K_{HH}$ Eq. (3.24b) | $K_{HH}$ Apsel & Luco |
|---|---|---|---|---|
| 0.00 | 3.92 | 5.51 | 3.21 | 4.62 |
| 0.25 | 4.67 | 6.40 | 5.06 | 6.32 |
| 0.50 | 5.50 | 7.09 | 6.44 | 7.51 |
| 1.00 | 7.41 | 8.33 | 7.99 | 9.52 |
| 2.00 | 9.79 | 10.47 | 10.29 | 12.86 |

## 4. RECTANGULAR REGIONS

Finally, we consider a rectangular soil island or truncated region of dimensions $2a_x \ x \ 2a_y \ x \ a_z$ $\left(-a_x < x < a_x, -a_y < y < a_y, 0 < z < a_z\right)$ carved out of the half-space $z > 0$. The boundary conditions of the planes $x = a_x$, $y = a_y$, and $z = a_z$ are approximated respectively by

$$\sigma_{xx} = -k_{xx}u_x \ , \qquad \sigma_{xy} = -k_{xy}u_y \ , \qquad \sigma_{xz} = -k_{xz}u_z \ , \qquad x = a_x \tag{4.1a}$$

$$\sigma_{yx} = -k_{yx}u_x \ , \qquad \sigma_{yy} = -k_{yy}u_y \ , \qquad \sigma_{yz} = -k_{yz}u_z \ , \qquad y = a_y \tag{4.1b}$$

$$\sigma_{zx} = -k_{zx}u_x \ , \qquad \sigma_{zy} = -k_{zy}u_y \ , \qquad \sigma_{zz} = -k_{zz}u_z \ , \qquad z = a_z \tag{4.1c}$$

where $k_{i\,j} = \left(i, j = x, y, z\right)$ represent the stiffness per unit area of boundary springs distributed over the artificial boundaries of the soil island. The notation for these springs is such that the first subscript (*i*) in $k_{i\,j}$ denotes the normal to the boundary plane on which the spring acts while the second subscript (*j*) denotes the direction of the spring.

The stiffnesses $\left(k_{xx}, k_{yx}, k_{zx}\right)$ of the springs acting in the x-direction on the x-, y-, and z-boundary planes are obtained in an approximate fashion from:

$$k_{xx}\left(a_x, y, z\right) = -\sigma_{xx}\left(a_x, y, z\right) / u_x\left(a_x, y, z\right) \tag{4.2a}$$

$$k_{yx}\left(x, a_y, z\right) = -\sigma_{yx}\left(x, a_y, z\right) / u_x\left(x, a_y, z\right) \tag{4.2b}$$

$$k_{zx}\left(x, y, a_z\right) = -\sigma_{zx}\left(x, y, a_z\right) / u_x\left(x, y, a_z\right) \tag{4.2c}$$

where the stresses and displacements in Eq. (4.2) correspond to those for a fundamental solution for an unbounded medium or a half-space leading to a significant displacement in the x-direction. For example, the solution of Kelvin (1848) for a concentrated horizontal force $P_x$ buried in a full space or Cerruti's (1882) solution for a concentrated horizontal force $P_x$ acting on the surface of a half-space should lead to useful asymptotic estimates for $k_{xx}$, $k_{yx}$, and $k_{zx}$ when the artificial boundary is sufficiently removed from the loaded area.

Equations similar to (4.2 a, b, c) are used to determine the stiffnesses ($k_{xy}$, $k_{yy}$, $k_{zy}$) and ($k_{xz}$, $k_{yz}$, $k_{zz}$) of distributed springs acting on the boundary planes in the y- and z-directions, respectively. The approximate boundary conditions on the planes $x = -a_x$ and $y = -a_y$ differ from those listed in Eqs. (4.1a) and (4.1b) only by the sign of the terms on the right-hand-side of these equations.

The results to be presented will show some spatial variation of the stiffness $k_{ij}$ over the boundary planes. Although it is clearly possible to use variable spring coefficients, in most applications it may be desirable to use a constant value over the face of the boundary planes. One way to obtain average spring coefficients per unit area $\bar{k}_{ij}$ is to use

$$\bar{k}_{ij} = -\left(\iint_{S_i} \sigma_{ij} dS_i \Big/ \iint_{S_i} u_j dS_i\right) \tag{4.3}$$

where $S_x = \{x = a_x,\ -a_y \le y \le a_y,\ 0 \le z \le a_z\}$, $S_y = \{-a_x \le x \le a_x,\ y = a_y,\ 0 \le z \le a_z\}$, and $S_z = \{-a_x \le x \le a_x,\ -a_y \le y \le a_y,\ z = a_z\}$. As in the case of Eq. (4.2), the stresses $\sigma_{ij}$ and displacements $u_j$ are taken to correspond to a fundamental solution.

### 4.1 BOUNDARY SPRING CONSTANTS BASED ON KELVIN'S SOLUTION

Initially, it is convenient to consider Kelvin's solution for a concentrated force in an unbounded full space (Thomson, 1882). In particular, for a concentrated force $P_x$ acting at the origin in the x-direction, it is found that

$$\left(\sigma_{xx}, \sigma_{yx}, \sigma_{zx}\right) = -\frac{P_x}{8\pi(1-\nu)R^3}\left[(1-2\nu) + 3\frac{x^2}{R^2}\right](x, y, z) \tag{4.4}$$

and

$$u_x = \frac{P_x}{16\pi\mu(1-\nu)R}\left[(3-4\nu) + \frac{x^2}{R^2}\right] \tag{4.5}$$

where $R = \left(x^2 + y^2 + z^2\right)^{1/2}$, $\mu$ is the shear modulus, and $\nu$ is Poisson's ratio. The corresponding expressions for the local spring constants $k_{xx}$, $k_{yx}$, and $k_{zx}$ are given by

$$\left(k_{xx}, k_{yx}, k_{zx}\right) = \frac{2\mu}{R^2}(x, y, z)\alpha_x \tag{4.6a}$$

where $\alpha_x$ is given by Eq. (4.7) below.

Proceeding in the same fashion for concentrated forces $P_y$ and $P_z$ acting in the x-and y-directions, leads to

$$\left(k_{xy},\, k_{yy},\, k_{zy}\right)=\frac{2\mu}{R^2}\left(x,y,z\right)\alpha_y \tag{4.6b}$$

and

$$\left(k_{xz},\, k_{yz},\, k_{zz}\right)=\frac{2\mu}{R^2}\left(x,y,z\right)\alpha_z \tag{4.6c}$$

where

$$\left(\alpha_x,\, \alpha_y,\, \alpha_z\,\right)=\left[\frac{3x^2+\left(1-2\nu\right)R^2}{x^2+\left(3-4\nu\right)R^2},\,\frac{3y^2+\left(1-2\nu\right)R^2}{y^2+\left(3-4\nu\right)R^2},\,\frac{3z^2+\left(1-2\nu\right)R^2}{z^2+\left(3-4\nu\right)R^2}\right] \tag{4.7}$$

Grouping the spring constants by the coordinate-plane on which they act, results in

$$\left(k_{xx},\, k_{xy},\, k_{xz}\right)=\frac{2\mu x}{R^2}\left(\alpha_x,\alpha_y,\alpha_z\right) \tag{4.8a}$$

$$\left(k_{yx},\, k_{yy},\, k_{yz}\right)=\frac{2\mu y}{R^2}\left(\alpha_x,\alpha_y,\alpha_z\right) \tag{4.8b}$$

$$\left(k_{zx},\, k_{zy},\, k_{zz}\right)=\frac{2\mu z}{R^2}\left(\alpha_x,\alpha_y,\alpha_z\right) \tag{4.8c}$$

It is instructive to consider the values of the spring constants at a few selected locations. First, on the plane $x=a_x$ at the points of coordinates $\left(a_x,0,0\right)$, $\left(a_x,\, a_x,\, 0\right)$, and $\left(a_x,0,a_x\right)$, the values of the spring constants $k_{xx}$, $k_{xy}$, and $k_{xz}$ are, respectively:

$$\left(k_{xx},\, k_{xy},\, k_{xz}\right)=\frac{2\mu}{a_x}\left(\frac{4-2\nu}{4-4\nu},\,\frac{1-2\nu}{3-4\nu},\,\frac{1-2\nu}{3-4\nu}\right) \tag{4.9a}$$

$$\left(k_{xx},\, k_{xy},\, k_{xz}\right)=\frac{\mu}{a_x}\left(\frac{5-4\nu}{7-8\nu},\,\frac{5-4\nu}{7-8\nu},\,\frac{1-2\nu}{3-4\nu}\right) \tag{4.9b}$$

and

$$\left(k_{xx},\, k_{xy},\, k_{xz}\right)=\frac{\mu}{a_x}\left(\frac{5-4\nu}{7-8\nu},\frac{1-2\nu}{3-4\nu},\,\frac{5-4\nu}{7-8\nu}\right) \tag{4.9c}$$

which for $\nu = 1/3$ become $\mu(2.5, 0.4, 0.4)/a_x$, $\mu(0.846, 0.846, 0.2)/a_x$, and $\mu(0.846, 0.2, 0.846)/a_x$, respectively. It should be noted that the value of $k_{xz}$ is not zero at $z = 0$ as Kelvin's solution pertains to a full space and does not satisfy the condition $\sigma_{xz} = 0$ at $z = 0$.

At selected points $(0,0,a_z)$, $(a_z,0,a_z)$, and $(0,a_z,a_z)$ all on the plane $z = a_z$, the corresponding values of the spring constants $k_{zx}$, $k_{zy}$, and $k_{zz}$ are, respectively,

$$\left(k_{zx}, k_{zy}, k_{zz}\right) = \frac{2\mu}{a_z}\left(\frac{1-2\nu}{3-4\nu}, \frac{1-2\nu}{3-4\nu}, \frac{4-2\nu}{4-4\nu}\right) \tag{4.10a}$$

$$\left(k_{zx}, k_{zy}, k_{zz}\right) = \frac{\mu}{a_z}\left(\frac{5-4\nu}{7-8\nu}, \frac{1-2\nu}{3-4\nu}, \frac{5-4\nu}{7-8\nu}\right) \tag{4.10b}$$

and,

$$\left(k_{zx}, k_{zy}, k_{zz}\right) = \frac{\mu}{a_z}\left(\frac{1-2\nu}{3-4\nu}, \frac{5-4\nu}{7-8\nu}, \frac{5-4\nu}{7-8\nu}\right) \tag{4.10c}$$

The variation of the normalized spring constants $a_x k_{xx}/\mu$, $a_y k_{yx}/\mu$, and $a_z k_{zx}/\mu$ over the planes $x = a_x$, $y = a_y$, and $z = a_z$ are shown in Figs. 4.1 a, b, c for the particular case $a_x = a_y = a_z = a$ and $\nu = 1/3$. The corresponding variation of the spring constants $a_x k_{xz}/\mu$, $a_y k_{yz}/\mu$, and $a_z k_{zz}/\mu$ over the planes $x = a_x$, $y = a_y$, and $z = a_z$ $\left(a_x = a_y = a_z = a\right)$ are shown in Figs. 4.1 d, e, f.

The average spring constants $\bar{k}_{ij}$ over the boundary planes obtained by use of Kelvin's solution for concentrated forces along the x-, y-, and z-direction are:

$$\left(\bar{k}_{xx}, \bar{k}_{xy}, \bar{k}_{xz}\right) = \frac{\mu}{a_x}\left(\beta_{xx}, \beta_{xy}, \beta_{xz}\right) \tag{4.11a}$$

$$\left(\bar{k}_{yx}, \bar{k}_{yy}, \bar{k}_{yz}\right) = \frac{\mu}{a_y}\left(\beta_{yx}, \beta_{yy}, \beta_{yz}\right) \tag{4.11b}$$

$$\left(\bar{k}_{zx}, \bar{k}_{zy}, \bar{k}_{zz}\right) = \frac{\mu}{a_z}\left(\beta_{zx}, \beta_{zy}, \beta_{zz}\right) \tag{4.11c}$$

where

$$\beta_{xx} = 2a_x^2 \frac{\iint_{S_x} \left[ (1-2\nu) + 3(x/R)^2 \right] R^{-3} dS_x}{\iint_{S_x} \left[ (3-4\nu) + (x/R)^2 \right] R^{-1} dS_x} \quad , \qquad x = a_x \tag{4.12a}$$

$$\beta_{xy} = 2a_x^2 \frac{\iint_{S_x} \left[ (1-2\nu) + 3(y/R)^2 \right] R^{-3} dS_x}{\iint_{S_x} \left[ (3-4\nu) + (y/R)^2 \right] R^{-1} dS_x} \quad , \qquad x = a_x \tag{4.12b}$$

$$\beta_{xz} = 2a_x^2 \frac{\iint_{S_x} \left[ (1-2\nu) + 3(z/R)^2 \right] R^{-3} dS_x}{\iint_{S_x} \left[ (3-4\nu) + (z/R)^2 \right] R^{-1} dS_x} \quad , \qquad x = a_x \quad . \tag{4.12c}$$

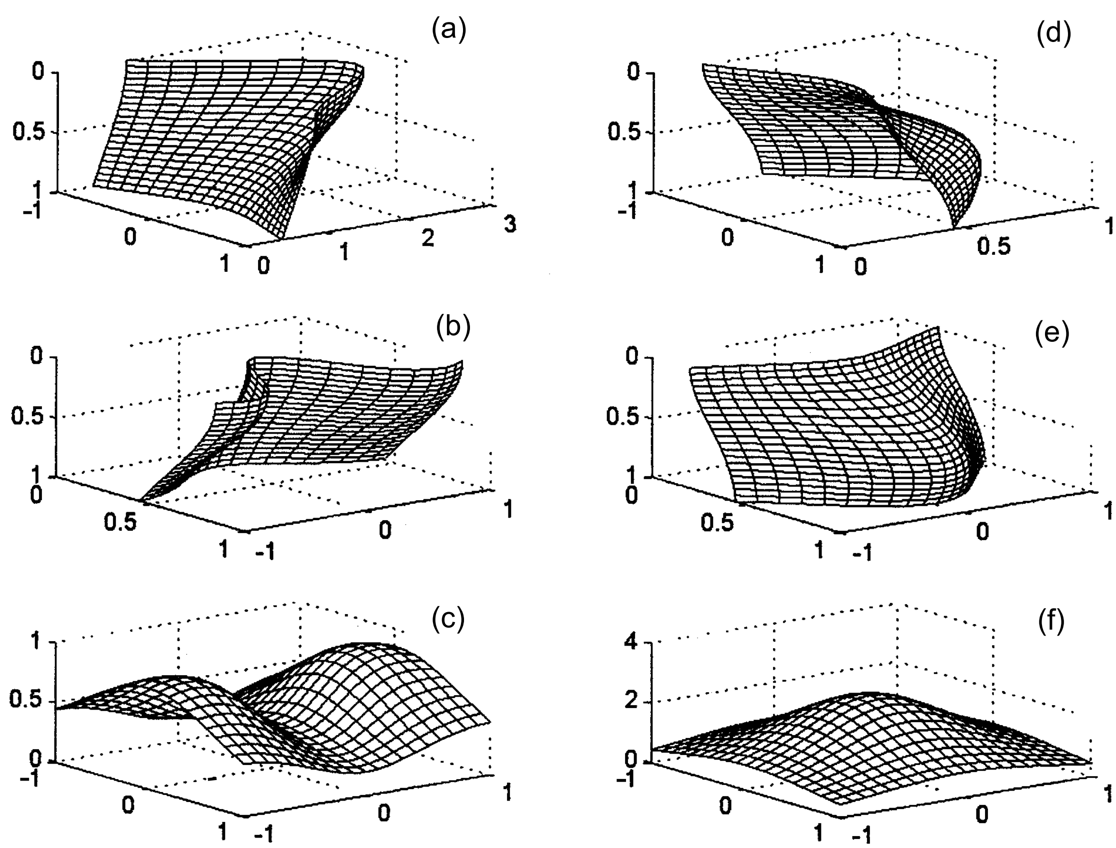


**Figure 4.1.** Variation of the normalized stiffness coefficients over the boundary planes for a rectangular soil island of dimensions $(2a \, x \, 2a \, x \, a)$. The figures correspond to: (a) $ak_{xx}(a,y,z)/\mu$, (b) $ak_{yx}(x,a,z)/\mu$, (c) $ak_{zx}(x,y,a)/\mu$, (d) $ak_{xz}(a,y,z)/\mu$, (e) $ak_{yz}(x,a,z)/\mu$, and (f) $ak_{zz}(x,y,a)/\mu$. Results are based on Kelvin's fundamental solution $(\nu = 1/3)$.

Similar expressions hold for the other normalized average spring constants $\beta_{ij}$. It should be noted that the normalized spring constants $\beta_{xx}$, $\beta_{xy}$, and $\beta_{xz}$ depend on $\nu$, $(a_z/a_x)$, and $(a_z/a_y)$. Similarly, $\beta_{yx}$, $\beta_{yy}$, and $\beta_{yz}$ depend only on $\nu$, $(a_z/a_x)$, and $(a_z/a_y)$. Finally, the coefficients $\beta_{zx}$, $\beta_{zy}$, $\beta_{zz}$ depend on $\nu$, $(a_z/a_x)$, and $(a_z/a_y)$.

Numerical values for the normalized average spring constants $\beta_{ij}$ for $\nu = 1/3$, $a_x = a_y = a$ and $a_z = h$ are presented in Table 4.1 as a function of $h/a$. The table includes values for $\beta_{xx} = \beta_{yy}$, $\beta_{xy} = \beta_{yx}$, $\beta_{xz} = \beta_{yz}$, $\beta_{zx} = \beta_{zy}$, and $\beta_{zz}$ where the equalities hold for the case $a_x = a_y$. The variations of these parameters with $h/a$ are shown in Fig. 4.2.

**Table 4.1.** Dimensionless Boundary Stiffness Coefficients for ($\nu = 1/3$, based on Kelvin's Solution) for a Rectangular Soil Island of Dimensions $2a\ x\ 2a\ x\ h$.

| $h/a$ | $\beta_{xx} = \beta_{yy}$ | $\beta_{xy} = \beta_{yx}$ | $\beta_{xz} = \beta_{yz}$ | $\beta_{zx} = \beta_{zy}$ | $\beta_{zz}$ |
|---|---|---|---|---|---|
| 0.0 | 1.841 | 0.717 | 0.321 | 0.000 | 0.000 |
| 0.2 | 1.809 | 0.707 | 0.346 | 0.207 | 0.293 |
| 0.4 | 1.723 | 0.680 | 0.406 | 0.370 | 0.596 |
| 0.6 | 1.610 | 0.642 | 0.474 | 0.478 | 0.890 |
| 0.8 | 1.492 | 0.602 | 0.528 | 0.537 | 1.155 |
| 1.0 | 1.382 | 0.562 | 0.562 | 0.562 | 1.382 |
| 1.2 | 1.286 | 0.526 | 0.579 | 0.567 | 1.569 |
| 1.4 | 1.204 | 0.494 | 0.584 | 0.560 | 1.721 |
| 1.6 | 1.134 | 0.466 | 0.580 | 0.549 | 1.844 |
| 1.8 | 1.075 | 0.442 | 0.572 | 0.536 | 1.943 |
| 2.0 | 1.024 | 0.421 | 0.560 | 0.523 | 2.023 |

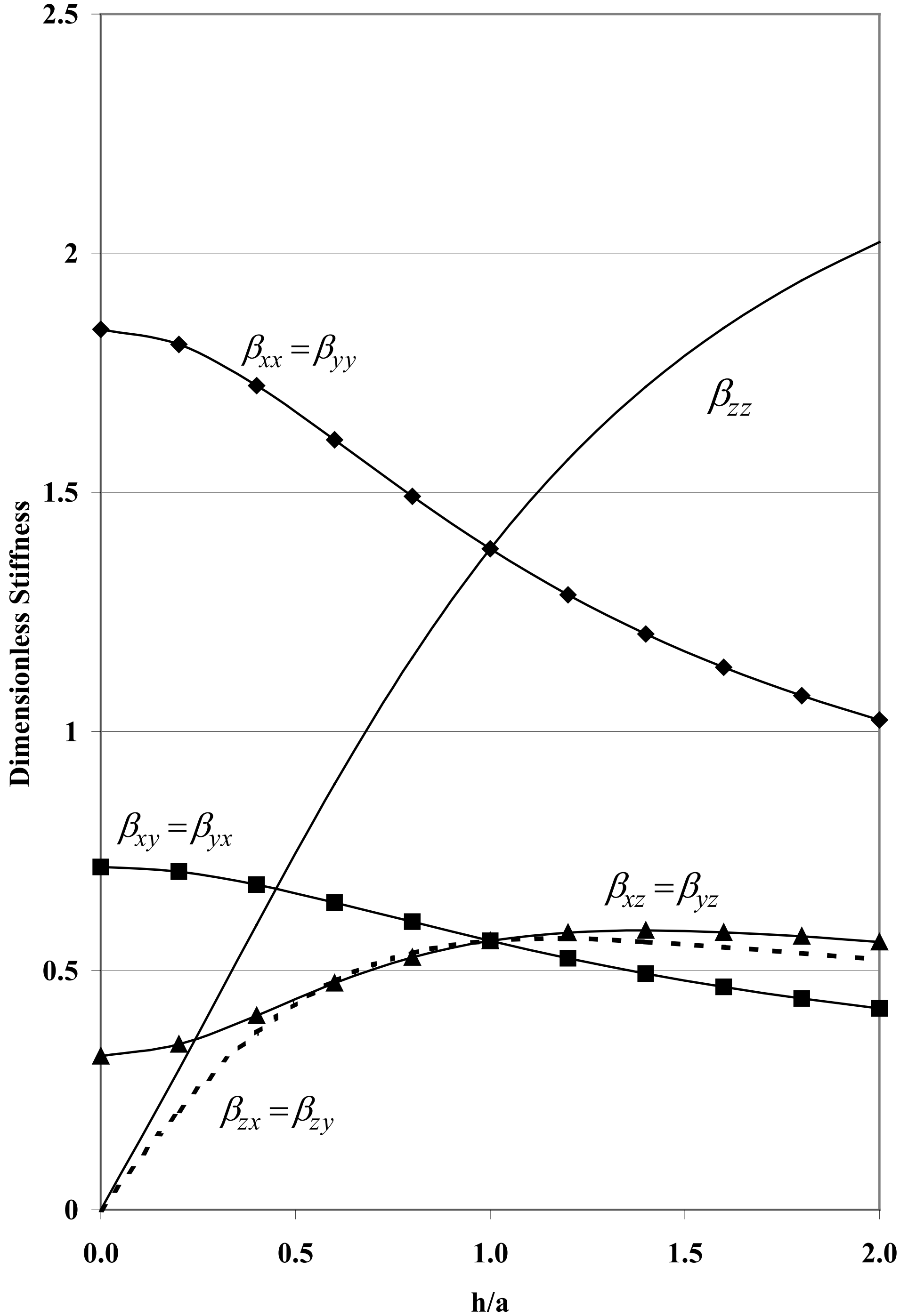


**Figure 4.2.** Normalized average stiffness coefficients for a rectangular soil island $(2a\,x\,2a\,x\,h)$. The numerical results are based on Kelvin's fundamental solution $(\nu = 1/3)$.

## 4.2 BOUNDARY SPRING CONSTANTS BASED ON BOUSSINESQ'S SOLUTION

The solution of Boussinesq (1878) for a concentrated normal load $P_z$ at the origin of an elastic half-space $(z > 0)$ can also be used to obtain approximate values for the distributed spring constants $k_{xz}$, $k_{yz}$, and $k_{zz}$. The relevant stress and displacement components are

$$\left(\sigma_{xz}, \sigma_{yz}, \sigma_{zz}\right) = -\frac{3P_z z^2}{2\pi R^5}(x, y, z) \tag{4.13}$$

and

$$u_z = \frac{P_z}{4\pi\mu R}\left[2(1-\nu)+\left(\frac{z}{R}\right)^2\right] \tag{4.14}$$

where $R = \left(x^2 + y^2 + z^2\right)^{1/2}$. The resulting distributed spring constants per unit area $k_{xz}$, $k_{yz}$, and $k_{zz}$ can be written in the form

$$\left(k_{xz}, k_{yz}, k_{zz}\right) = \frac{2\mu}{R^2}(x, y, z)\alpha_z \tag{4.15}$$

where

$$\alpha_z = \left[\frac{3z^2}{z^2 + 2(1-\nu)R^2}\right] \tag{4.16}$$

The normal spring constant $k_{zz}$ takes the values: $k_{zz} = \left[6/(3-2\nu)\right](\mu/a_z)$, $k_{zz} = \left[3/(5-4\nu)\right](\mu/a_z)$, $k_{zz} = \left[3/(5-4\nu)\right](\mu/a_z)$, respectively, at the points $(0,0,a_z)$, $(a_z,0,a_z)$, and $(0,a_z,a_z)$ on the $z = a_z$ plane. These values are very similar to those obtained on the basis of Kelvin's solution.

The shear spring constant $k_{xz}$ is zero at all points $z = 0$. At the point $(a_x,0,a_x)$, it takes the value $k_{xz} = \left[3/(5-4\nu)\right](\mu/a_x)$, which is not very different from the value $\left[(5-4\nu)/(7-8\nu)\right](\mu/a_x)$ obtained from Kelvin's solution.

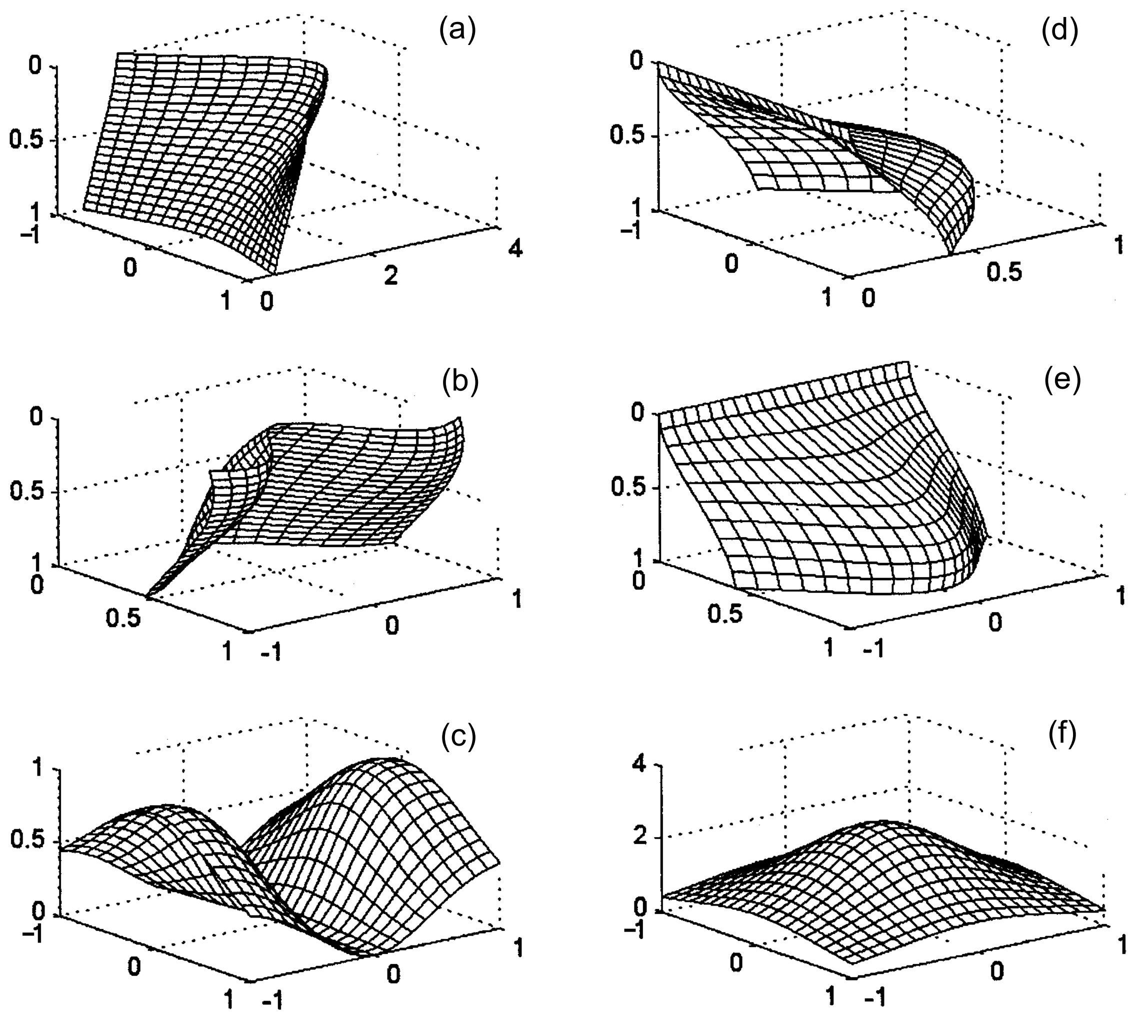


**Figure 4.3.** Variation of the normalized stiffness coefficients over the boundary planes for a rectangular soil island of dimensions $(2a\,x\,2a\,x\,a)$. The figures correspond to: (a) $ak_{xx}(a,y,z)/\mu$, (b) $ak_{yx}(x,a,z)/\mu$, (c) $ak_{zx}(x,y,a)/\mu$, (d) $ak_{xz}(a,y,z)/\mu$, (e) $ak_{yz}(x,a,z)/\mu$, and (f) $ak_{zz}(x,y,a)/\mu$. Results are based on Cerruti's (a, b, c) and Boussinesq's (d, e, f) fundamental solutions $(\nu = 1/3)$.

The variations of the normalized spring constants $a_x k_{xz}/\mu$, $a_y k_{yz}/\mu$, and $a_z k_{zz}/\mu$ over the planes $x = a_x$, $y = a_y$, and $z = a_z$ are shown in Figs 4.3 d, e, f, respectively, for the case $a_x = a_y = a_z = a$ and $\nu = 1/3$. Again, although the spatial variation is significant, it is convenient to introduce some uniform spring constants calculated by use of Eq. (4.3). The resulting expressions for $\bar{k}_{xz}$, $\bar{k}_{yz}$, and $\bar{k}_{zz}$ are given by

$$\left(\bar{k}_{xz}, \bar{k}_{yz}, \bar{k}_{zz}\right) = \mu\left(\beta_{xz}/a_x, \beta_{yz}/a_y, \beta_{zz}/a_z\right) \tag{4.17}$$

where

$$\beta_{xz} = 6a_x^2 \frac{\iint_{S_x} (z/R)^2 R^{-3} dS_x}{\iint_{S_x} \left[2(1-\nu)+(z/R)^2\right] R^{-1} dS_x} \quad , \qquad (x = a_x) \tag{4.18a}$$

$$\beta_{yz} = 6a_y^2 \frac{\iint_{S_y} (z/R)^2 R^{-3} dS_y}{\iint_{S_y} \left[2(1-\nu)+(z/R)^2\right] R^{-1} dS_y} \quad , \qquad (y = a_y) \tag{4.18b}$$

$$\beta_{zz} = 6a_z^2 \frac{\iint_{S_z} (z/R)^2 R^{-3} dS_z}{\iint_{S_z} \left[2(1-\nu)+(z/R)^2\right] R^{-1} dS_z} \quad , \qquad (z = a_z) \tag{4.18c}$$

**Table 4.2.** Dimensionless Boundary Stiffness Coefficients for a Rectangular Soil Island of Dimensions $2a \ x \ 2a \ x \ h$. The results are based on the solutions of Boussinesq and Cerruti $(\nu = 1/3)$.

| $h/a$ | $\beta_{xx} = \beta_{yy}$ | $\beta_{xy} = \beta_{yx}$ | $\beta_{xz} = \beta_{yz}$ | $\beta_{zx} = \beta_{zy}$ | $\beta_{zz}$ |
|---|---|---|---|---|---|
| 0.0 | 2.004 | 0.729 | 0.000 | 0.000 | 0.000 |
| 0.2 | 1.966 | 0.706 | 0.038 | 0.199 | 0.281 |
| 0.4 | 1.866 | 0.670 | 0.131 | 0.345 | 0.582 |
| 0.6 | 1.734 | 0.626 | 0.238 | 0.427 | 0.881 |
| 0.8 | 1.599 | 0.581 | 0.329 | 0.453 | 1.156 |
| 1.0 | 1.474 | 0.539 | 0.394 | 0.443 | 1.392 |
| 1.2 | 1.366 | 0.501 | 0.435 | 0.413 | 1.589 |
| 1.4 | 1.275 | 0.468 | 0.458 | 0.376 | 1.749 |
| 1.6 | 1.198 | 0.440 | 0.468 | 0.337 | 1.878 |
| 1.8 | 1.134 | 0.416 | 0.470 | 0.300 | 1.982 |
| 2.0 | 1.079 | 0.396 | 0.467 | 0.266 | 2.067 |

Numerical values for the normalized uniform spring constants $\beta_{xz}$, $\beta_{yz}$, $\beta_{zz}$ for $\nu = 1/3$ and $a_x = a_y = a$ and $a_z = h$ are presented in Table 4.2 as a function of $h/a$. The variations of these spring constants with $h/a$ are shown in Fig. 4.4.

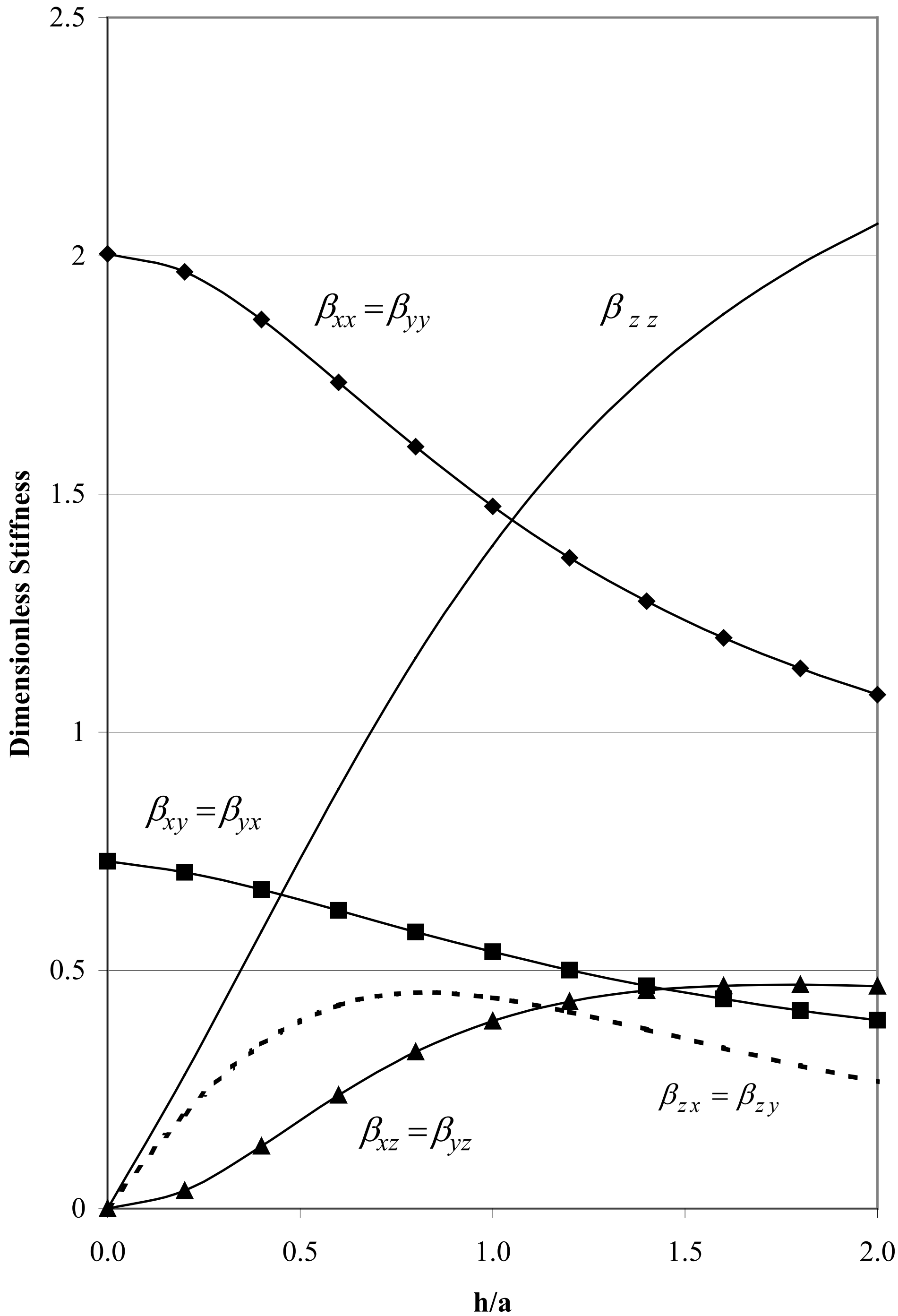


**Figure 4.4.** Normalized average stiffness coefficients for a rectangular soil island $(2a\,x\,2a\,x\,h)$. The numerical results are based on the fundamental solutions of Boussinesq and Cerruti $(\nu = 1/3)$.

## 4.3 BOUNDARY SPRING CONSTANTS BASED ON CERRUTI'S SOLUTION

Cerruti's (1882) solution for a concentrated load $P_x$ tangential to the boundary plane of an elastic half-space $\left(z>0\right)$ can be used to determine alternative estimates for the spring constants $k_{xx}$, $k_{yx}$, $k_{zx}$. The stress and displacement components of interest are:

$$\sigma_{xx} = -\frac{P_x x}{2\pi R^3}\left[\frac{3x^2}{R^2} - \frac{\left(1-2\nu\right)}{\left(R+z\right)^2}\left(x^2+z^2-\frac{2Ry^2}{R+z}\right)\right] \tag{4.19a}$$

$$\sigma_{yx} = -\frac{P_x y}{2\pi R^3}\left[\frac{3x^2}{R^2} + \frac{\left(1-2\nu\right)}{\left(R+z\right)^2}\left(y^2+z^2-\frac{2Rx^2}{R+z}\right)\right] \tag{4.19b}$$

$$\sigma_{zx} = -\frac{P_x z}{2\pi R^3}\left[\frac{3x^2}{R^2}\right] \tag{4.19c}$$

and

$$u_x = \frac{P_x}{4\pi\mu R}\left\{1+\frac{x^2}{R^2}+\left(1-2\nu\right)\left[\frac{R}{\left(R+z\right)}-\frac{x^2}{\left(R+z\right)^2}\right]\right\} \tag{4.20}$$

where $R=\left(x^2+y^2+z^2\right)^{1/2}$. The resulting spring constants are given by

$$\left(k_{xx}, k_{yx}, k_{zx}\right) = \frac{2\mu}{R^2}\left(x\alpha_{xx}, y\alpha_{yx}, z\alpha_{zx}\right) \tag{4.21}$$

where

$$\alpha_{xx} = \frac{\left\{\frac{3x^2}{R^2} - \frac{\left(1-2\nu\right)}{\left(R+z\right)^2}\left[x^2+z^2-\frac{2Ry^2}{R+z}\right]\right\}}{\left\{1+\frac{x^2}{R^2}+\left(1-2\nu\right)\left[\frac{R}{R+z}-\frac{x^2}{\left(R+z\right)^2}\right]\right\}} \tag{4.22a}$$

$$\alpha_{yx} = \frac{\left\{\frac{3x^2}{R^2} + \frac{\left(1-2\nu\right)}{\left(R+z\right)^2}\left[y^2+z^2-\frac{2Rx^2}{R+z}\right]\right\}}{\left\{1+\frac{x^2}{R^2}+\left(1-2\nu\right)\left[\frac{R}{R+z}-\frac{x^2}{\left(R+z\right)^2}\right]\right\}} \tag{4.22b}$$

$$\alpha_{zx} = \frac{\left(\dfrac{3x^2}{R^2}\right)}{\left\{1 + \dfrac{x^2}{R^2} + (1-2\nu)\left[\dfrac{R}{R+z} - \dfrac{x^2}{(R+z)^2}\right]\right\}} \tag{4.22c}$$

It should be noted that for $\nu = 1/2$, $\alpha_{xx} = \alpha_{yx} = \alpha_{zx} = \alpha_x$ where $\alpha_x$ is given by Eq. (4.7).

The values of the spring constant $k_{xx}$ at the selected points $(a_x, 0, 0)$ and $(a_x, a_x, 0)$ on the $x = a_x$ plane are $k_{xx} = 2(1+\nu)\mu/a_x$ and $k_{xx} = \mu/a_x$, respectively. For $\nu = 1/3$, these values are $2.67\mu/a_x$ and $\mu/a_x$ which should be compared with $2.5\mu/a_x$ and $0.846\mu/a_x$ for the constants based on Kelvin's solution. At the point of coordinates $(a_x, 0, a_x)$ and for $\nu = 1/3$, the value $k_{xx} = 0.846\mu/a_x$ coincides with the results based on Kelvin's solution.

On the plane $y = a_y$, the values of the spring constant $k_{yx}$ at the points $(0, a_y, 0)$ and $(a_y, a_y, 0)$ are $k_{yx} = \left[(1-2\nu)/(1-\nu)\right]\mu/a_y$ and $k_{yx} = \left[(1+\nu)/(2-\nu)\right]\mu/a_y$, respectively. For $\nu = 1/3$, these values correspond to $0.5\mu/a_y$ and $0.8\mu/a_y$, respectively, which should be compared with $0.4\mu/a_y$ and $0.846\mu/a_y$ for the springs based on Kelvin's solution. For $\nu = 1/3$, the value of $k_{yx}$ at the point $(0, a_y, a_y)$ is $k_{yx} = 0.0957\mu/a_y$ which should be compared with $0.2\mu/a_y$ in Kelvin's solution.

Finally, on the plane $z = a_z$, the coefficients $k_{zx}$ are zero at all points on $x = 0$, and $k_{zx} = \left[1 + \frac{2}{3}\frac{(1-2\nu)}{1+\sqrt{2}}\right]^{-1} \mu a_z^{-1}$ at the point $(a_z, 0, a_z)$. For $\nu = 1/3$, this value corresponds to $0.916\mu/a_z$ which should be compared with $0.846\mu/a_z$ in Kelvin's solution.

The variation of the normalized spring constants $a_x k_{xx}/\mu$, $a_y k_{yx}/\mu$, $a_z k_{zx}/\mu$ over the planes $x = a_x$, $y = a_y$, and $z = a_z$ are shown in Figures 4.3 a, b, c for the case $\nu = 1/3$ and $a_x = a_y = a_z = a$. The most significant difference with the results in Figures 4.1 a, b, c is that $k_{zx}$ is zero for $x = 0$.

Representative uniform values of the spring constants $\left(\bar{k}_{xx}, \bar{k}_{yx}, \bar{k}_{zx}\right)$ over the faces of the boundary planes can be calculated on the basis of Eq. (4.3). The resulting expressions are

$$\left(\bar{k}_{xx}, \bar{k}_{yx}, \bar{k}_{zx}\right) = \mu\left(\beta_{xx}/a_x, \beta_{yx}/a_y, \beta_{zx}/a_z\right) \tag{4.23}$$

where

$$\beta_{xx} = 2a_x^2 \frac{\iint_{S_x} \left\{ \frac{3x^2}{R^2} - \frac{(1-2\nu)}{(R+z)^2}\left[x^2 + z^2 - \frac{2R_y^2}{R+z}\right]\right\} R^{-3} dS_x}{\iint_{S_x} \left\{1 + \frac{x^2}{R^2} + (1-2\nu)\left[\frac{R}{(R+z)} - \frac{x^2}{(R+z)^2}\right]\right\} R^{-1} dS_x}, \qquad (x = a_x) \tag{4.24a}$$

$$\beta_{yx} = 2a_y^2 \frac{\iint_{S_y} \left\{ \frac{3x^2}{R^2} + \frac{(1-2\nu)}{(R+z)^2}\left[y^2 + z^2 - \frac{2R_y^2}{R+z}\right]\right\} R^{-3} dS_y}{\iint_{S_y} \left\{1 + \frac{x^2}{R^2} + (1-2\nu)\left[\frac{R}{(R+z)} - \frac{x^2}{(R+z)^2}\right]\right\} R^{-1} dS_y}, \qquad (y = a_y) \tag{4.24b}$$

$$\beta_{zx} = 2a_z^2 \frac{\iint_{S_z} \left(\frac{3x^2}{R^2}\right) R^{-3} dS_z}{\iint_{S_z} \left\{1 + \frac{x^2}{R^2} + (1-2\nu)\left[\frac{R}{(R+z)} - \frac{x^2}{(R+z)^2}\right]\right\} R^{-1} dS_z}, \qquad (z = a_z) \tag{4.24c}$$

Numerical values for the coefficients $\beta_{xx}$, $\beta_{yx}$, $\beta_{zx}$, are presented in Table 4.2 for the case $\nu = 1/3$, $a_x = a_y = a$ and $a_z = h$. The results are presented as a function of $h/a$. The variations of these parameters with $h/a$ are shown in Fig. 4.4. The most noticeable difference with respect to the results in Fig. 4.2 is that the coefficient $\beta_{zx}$ tends to zero as $h/a$ increases while $\beta_{zx}$ based on Kelvin's solution tends to 0.4.

### 4.4 COMPARISONS FOR A RIGID FOUNDATION

A severe test of the spring constants obtained here, results from considering the static force-displacement relation for a rigid square foundation of area $(2a \times 2a)$ embedded to a depth $h$ in a uniform elastic half-space. Placing the boundary springs in direct contact with the foundation $(a_x = a,\ a_y = a,\ a_z = h)$ results in the total vertical $(K_{VV})$ and horizontal $(K_{HH})$ stiffness coefficients:

$$K_{VV} = (2a \times 2a)\frac{\mu}{h}\beta_{zz} + 4(2a \times h)\frac{\mu}{a}\beta_{xz} \tag{4.25a}$$

$$K_{VV} = \mu a\left[4\left(\frac{a}{h}\right)\beta_{zz} + 8\left(\frac{h}{a}\right)\beta_{xz}\right] \tag{4.25b}$$

and

$$K_{HH} = (2a \times 2a)\frac{\mu}{h}\beta_{zx} + 2(2a \times h)\frac{\mu}{a}\beta_{yx} + 2(2a \times h)\frac{\mu}{a}\beta_{xx} \tag{4.26a}$$

$$K_{HH} = \mu a\left[4\left(\frac{a}{h}\right)\beta_{zx} + 4\left(\frac{h}{a}\right)\beta_{yx} + 4\left(\frac{h}{a}\right)\beta_{xx}\right] \tag{4.26b}$$

The approximate results based on Eqs. (4.25b) and (4.26b) and on the average spring constants based on Kelvin's, Boussinesq's, and Cerruti's solutions are compared in Table 4.3 with more accurate results obtained by Mita and Luco (1989) for a rigid square foundation $(2a \times 2a)$ embedded to a depth $h$ in a uniform half-space $(\nu = 1/3)$.

**Table 4.3**. Comparison of Impedance Functions for a Rigid Rectangular Foundation $(2a\,x\,2a\,x\,h)$ Embedded in an Elastic Half-Space $(\nu = 1/3)$.

| *h/a* | $K_{VV}$ *(1)* | $K_{VV}$ *(2)* | $K_{VV}$ *Mita & Luco* | $K_{HH}$ *(3)* | $K_{HH}$ *(4)* | $K_{HH}$ *Mita & Luco* |
|---|---|---|---|---|---|---|
| 0.0 | 5.36 | 5.74 | 6.96 | 4.38 | 4.48 | 5.51 |
| 0.5 | 6.61 | 7.73 | 8.58 | 8.05 | 8.11 | 8.66 |
| 1.0 | 8.72 | 10.02 | 9.93 | 9.82 | 10.02 | 10.87 |
| 1.5 | 10.41 | 11.76 | 11.13 | 11.08 | 11.36 | 12.79 |

(1) based on Boussinesq's fundamental solution

(2) and (4) based on Kelvin's fundamental solution

(3) based on Cerruti's fundamental solution

The estimate of the vertical impedance $K_{VV}$ based on Boussinesq's solution is systematically lower than the results based on Kelvin's solution, and also lower than the more exact results of Mita and Luco (1989). The differences are still less than 23 percent in this extreme case in which the boundary springs are attached directly to the foundation. The comparisons for the horizontal impedance $K_{HH}$ indicate that the results based on Cerruti's solution are slightly lower than those based on Kelvin's solution but still approximate the actual stiffness coefficient with an error of less than 21 percent.

## 5. CONCLUSIONS

Simple, approximate expressions for the stiffness coefficients per unit area of boundary springs distributed over the artificial boundary of truncated elastic regions have been obtained. These springs serve as approximate boundary conditions on the artificial boundary so that the solution within the truncated region approaches that for an unbounded half-space. The springs have been selected to match asymptotically the solutions for concentrated forces located within the truncated region and away from the artificial boundary.

Numerical values for the stiffnesses of the boundary springs have been presented for hemispherical, cylindrical, and rectangular regions. The results have been tested by applying these springs directly to the boundary of rigid foundations and by comparing the resulting static impedance functions with those for the foundations on an elastic half-space. Although these springs have not been formulated to be applied directly to the foundation, but rather to a layer soil island surrounding the foundation, the differences between the two sets of foundation impedance functions for translational motion of the foundation amount to less than 30 percent. As expected, the differences for the rotational degrees of freedom are much larger when the springs are applied directly to the foundation. However, since the response to moments is more localized than that to forces, it is expected that a moderately sized soil island or truncated region resting on the boundary springs obtained here will be sufficient to accurately represent the translational and rotational response components at points away from the artificial boundary.

The static springs presented herein could be used in conjunction with Lysmer's dampers to obtain a simple, approximate non-reflecting boundary for dynamic soil-structure interaction problems.

# 6. ACKNOWLEDGMENTS

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